COMPLEX GEOMETRICAL OPTICS SOLUTIONS

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The so-called *complex geometric optics (CGO) solutions* were introduced in the context of inverse problems and imaging by Sylvester and Uhlmann in 1987 [9], reinventing and extending the quantum scattering results of Faddeev [6]. CGO solutions are useful perturbations of the exponential function, allowing the construction of problem-specific nonlinear Fourier analyses. The first numerical computations of CGO solutions were published in [8].

Astala and Päivärinta used novel CGO solutions for the Beltrami equation in their solution [3] of the two-dimensional version of the inverse conductivity problem stated by Calderón in 1980 [4]. Assume given a strictly positive conductivity function $\sigma \in L^\infty(\Omega)$, where $\Omega \subset \mathbb{R}^2$ is the unit disc. Applying a voltage potential $g : \partial\Omega \to \mathbb{R}$ at the boundary results in a potential u inside Ω satisfying

$$\nabla \cdot (\sigma(z)\nabla u(z)) = 0 \text{ in } \Omega, \qquad u|_{\partial\Omega} = f.$$

The goal in Calderón's problem is to recover the inner electric conductivity $\sigma(z)$ from voltage-to-current measurements $f \mapsto \sigma\frac{\partial u}{\partial \vec{n}}$. See [5] for related medical imaging applications.

Define $\mu = \frac{1-\sigma}{1+\sigma}$ and denote $z = x + iy$ and $\overline{\partial}_z := \frac{1}{2}(\partial_x + i\partial_y)$. It was shown in [3] that the Beltrami equation

$$\overline{\partial}_z f_\mu = \mu\overline{\partial_z f_\mu}$$

has a unique CGO solution f_μ of the following form:

$$f_\mu(z,k) = e^{ikz}(1 + \omega_\mu(z,k)), \quad \text{with} \quad \omega_\mu(z,k) = \mathcal{O}\left(\frac{1}{z}\right) \text{ as } |z| \to \infty.$$

Here $k \in \mathbb{C}$ is a complex parameter that can be thought of as a generalized frequency-domain variable.

Numerical evaluation of the Beltrami-type CGO solution f_μ was first introduced in [2], speeded-up in [7], and applied to electrical impedance imaging in [1].

To illustrate a CGO solution numerically, we construct a simulated phantom modelling roughly a cross-section of a human chest. See the left panel in Figure 1. We take $k = 10$ and compute the the asymptotic

Date: July 27, 2012.

part $\omega_\mu(z,10)$ of the complex geometric optics solution $f_\mu(z,10)$ for z ranging in the unit disc. We plot the complex-valued function $\omega_\mu(\,\cdot\,,10)$ using the phase-plot method discussed in [10]. See the right panel in Figure 1 for the result.

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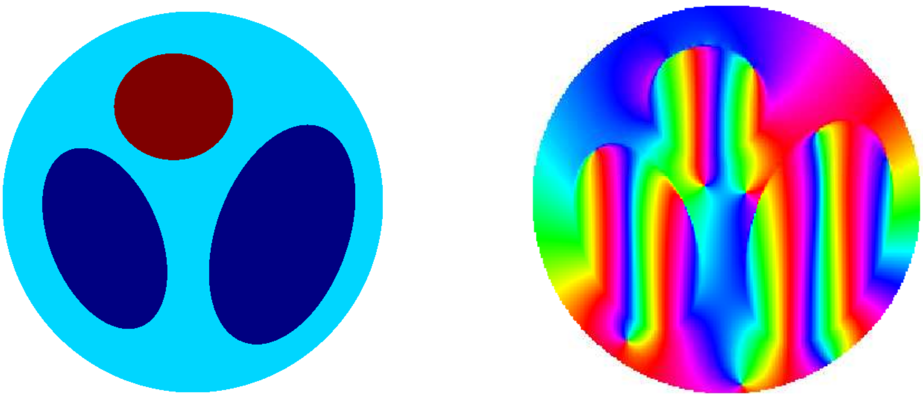

Figure 1. Left: Conductivity distribution in the unit disc. Dark blue color denotes lungs filled with air with conductivity 0.5. Red color denotes the heart filled with blood with conductivity 2. Light blue is the background conductivity equal to one. Right: the asymptotic part $\omega_\mu(\,\cdot\,, 10)$ of the complex geometric optics solution f_μ.