A Statistical Generalization of the Transformed Eulerian Mean Circulation

Olivier Pauluis (Courant Institute/NYU)
Tiffany Shaw (Columbia University)
Frederic Laliberte (U. of Toronto)

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How to describe the circulation?

- This requires some averaging, usual both in time and longitude.
- The circulation can be diagnosed by computing the stream function:

\[ \Psi(\varphi, p) = \int_{p}^{p_{surf}} 2\pi \bar{v} a \cos \varphi \frac{dp}{g} \]
Eulerian-mean circulation exhibits the ‘classic’ three-cell structure. But the Ferrel cell is a reverse circulation that transports energy toward the equator.
• **BUT** the mean meridional circulation depends very strongly on the vertical coordinate which is used for the averaging.

‘dry’ circulation averaged on surfaces of constant potential temperature

‘moist’ circulation averaged on surfaces of constant equivalent potential temperature

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from Pauluis et al. (2010)
Why the circulation in eulerian and isentropic coordinates are in the opposite direction?

- In the midlatitudes, the flow is highly turbulent: the meridional velocity alternates between poleward and equatorward at all levels.

\[ v < 0 \] \hspace{2cm} \[ v > 0 \]

longitude
In the stormtracks: Eulerian-mean circulation

- In the midlatitudes, the flow is highly turbulent: the meridional velocity alternates between poleward and equatorward at all levels.
- This idealized eddies is associated with a poleward flow at high pressure/low level, and equatorward flow at high level

\[ \bar{v}^p < 0 \text{ at low pressure} \]

\[ \bar{v}^p > 0 \text{ at high pressure} \]

Isobaric surface
Thickness variations are such that the upper isentropic layer encompass larger fraction of the poleward flow. Such pattern also corresponds to a net poleward energy mass transport.
• Moist isentropes found in the upper troposphere also intersects the Earth’s surface.
• Such situation corresponds to a poleward flow of warm, moist air near the surface.
The global circulation has two poleward components in the midlatitudes:
- an upper tropospheric branch of high $\theta_e - \theta_l$;
- an a lower branch of warm, most air with high $\theta_e$-low $\theta_l$, which ascent into the upper troposphere within the stormtracks.

Mass transport is comparable in each branch.
• The streamfunction in an arbitrary coordinate can be defined as

\[ \Psi_\zeta(\phi, \zeta_0) = \frac{1}{T} \int_0^T \int_0^{p_{surf}} \int_0^{2\pi} \frac{a \cos \phi}{g} vH(\zeta_0 - \zeta(\lambda, \phi, p, t)) \, d\lambda \, dp \, dt \]

• It is straightforward to compute given 4 dimensional data.

• How to recover it if we are only given the time mean circulation and eddy statistics?

\[ \overline{v}, \overline{\zeta}, \overline{v'^2}, \overline{v'\zeta'}, \overline{\zeta'^2}(\phi, p) \quad ? \quad \Phi_\zeta(\phi, \zeta) \]
Transformed Eulerian Mean (TEM) Circulation

\[ p' = -\left( \frac{\partial \tilde{\zeta}}{\partial p} \right)^{-1} \zeta' \]

**TEM residual circulation:**

\[ \Psi_\zeta(\phi, \tilde{\zeta}(p)) = \int_0^p \bar{v} dp - \left( \frac{\partial \tilde{\zeta}}{\partial p} \right)^{-1} \bar{v'} \zeta' + \text{h.o.t} \]

- Eulerian-mean circulation
- Eddy component
Transformed Eulerian Mean (TEM) Circulation

TEM residual circulation:

$$p' = -\left(\frac{\partial \bar{\zeta}}{\partial p}\right)^{-1} \zeta'$$

Great, but TEM breaks down when $\bar{\zeta}(p)$ is not stratified

TEM residual circulation:

$$\Psi_\zeta(\phi, \bar{\zeta}(p)) = \int_0^p \bar{v} dp - \left(\frac{\partial \bar{\zeta}}{\partial p}\right)^{-1} v' \zeta' + \text{h.o.t}$$

Eulerian-mean circulation

Eddy component
The Statistical Transformed Eulerian Mean (STEM) Circulation

We introduce a joint distribution of the meridional mass transport $m(\phi, p, \zeta)$ so that the streamfunction can be written as

$$\Psi_\zeta(\phi, \zeta) = \int_{-\infty}^{\zeta} \int_{0}^{P_{surf}} m(\phi, p, \zeta) dp d\zeta$$

We assume that at each pressure and latitude, the velocity and $\zeta$ obey a Gaussian distribution

$$f(v, \zeta) = \frac{1}{2\pi|\Sigma|^{1/2}} \exp \left( -\frac{1}{2} (v', \zeta')^T \Sigma^{-1} (v', \zeta') \right)$$

with the covariance matrix

$$\Sigma = \begin{pmatrix} \overline{v'^2} & \overline{v'\zeta'} \\ \overline{v'\zeta'} & \overline{\zeta'^2} \end{pmatrix}$$
Under these assumptions, the mean velocity for a given value of $\zeta$ is

$$E(v|\zeta) = \bar{v} + \frac{v'\zeta'}{\zeta r^2}(\zeta - \bar{\zeta})$$

The joint distribution follows:

$$m(\phi, p, \zeta) = \frac{2\pi a \cos \phi}{g} E(v|\zeta) \mathcal{N}_Z(\bar{\zeta}, \bar{\zeta} r^2)$$

$$= m_{\text{mean}}(\phi, p, \zeta) + m_{\text{eddy}}(\phi, p, \zeta)$$

with

$$m_{\text{mean}}(\phi, p, \zeta) = \frac{\sqrt{2\pi a \cos \phi}}{g \bar{\zeta}^{1/2}} \bar{v} \exp \left(\frac{-(\zeta - \bar{\zeta})^2}{2\bar{\zeta} r^2}\right)$$

$$m_{\text{eddy}}(\phi, p, \zeta) = \frac{\sqrt{2\pi a \cos \phi}}{g \bar{\zeta}^{3/2}} v'\zeta' (\zeta - \bar{\zeta}) \exp \left(\frac{-(\zeta - \bar{\zeta})^2}{2\bar{\zeta} r^2}\right)$$
\[ m_{\text{mean}}(\phi, p, \zeta) = \frac{\sqrt{2\pi a \cos \phi}}{g \zeta r^2^{1/2}} \bar{v} \exp\left(\frac{-(\zeta - \bar{\zeta})^2}{2\zeta r^2}\right) \]

\[ m_{\text{eddy}}(\phi, p, \zeta) = \frac{\sqrt{2\pi a \cos \phi}}{g \zeta r^2^{3/2}} \bar{v}' \zeta'(\zeta - \bar{\zeta}) \exp\left(\frac{-(\zeta - \bar{\zeta})^2}{2\zeta r^2}\right) \]
\[ \Psi_{mean}(\phi, \zeta) = \frac{\pi a \cos \phi}{g} \int_0^{\infty} \overline{v} \left(1 + \text{erf} \left( \frac{\zeta - \zeta}{\sqrt{2\zeta^2}} \right) \right) \, dp \]  

\[ \Psi_{eddy}(\phi, \zeta) = \frac{-2\pi a \cos \phi}{g} \int_0^{\infty} \frac{1}{\sqrt{2\pi \zeta^2}} \overline{v'} \zeta' \exp \left( \frac{-\left(\zeta - \overline{\zeta}\right)^2}{2\zeta^2} \right) \, dp \]
\[ \Psi_{\theta_e, \text{mean}}^{\text{DJF}} \]

\[ \Psi_{\theta_e, \text{eddy}}^{\text{DJF}} \]

\[ \Psi_{\theta_e, \text{STEM}}^{\text{DJF}} \]

\[ \Psi_{\theta_e}^{\text{DJF}} \]
\[ \frac{v'\theta_e'}{c_p} \approx \frac{L}{v'} \frac{v'q'}{q'}, \quad \Psi_{\theta_e,eddys} = \Psi_{\theta_e,SH} + \Psi_{\theta_e,LH} \]
Sensible heat contribution peaks in the midlatitudes
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Latent heat contribution peaks on the equatorward side of the stormtracks
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Latent heat contribution extends far in the tropics


Sensible heat contribution peaks in the midlatitudes

Latent heat contribution peaks on the equatorward side of the stormtracks

Both contributions overlap across the stormtracks, so that the total circulation is larger than either component.

Latent heat contribution extends far in the tropics
Relationship between TEM and STEM

• It can be shown formally that as the variance $\overline{\zeta'^2}$ goes to 0, the STEM streamfunction converges toward the TEM, i.e:

$$\lim_{\overline{\zeta'^2} \to 0} \Psi_{mean}(\phi, \overline{\zeta}(p)) = \Psi_{TEM,mean}(\phi, p)$$

$$\lim_{\overline{\zeta'^2} \to 0} \Psi_{eddy}(\phi, \overline{\zeta}(p)) = \left( \frac{\partial \overline{\zeta}}{\partial p} \right)^{-1} \overline{u'\zeta'}(\phi, p) = \Psi_{TEM,eddy}(p, \phi)$$
$\psi_{\theta,STEM}$ DJF variance = 64 K$^2$

$\psi_{\theta,STEM}$ DJF variance = 16 K$^2$

$\psi_{\theta,STEM}$ DJF variance = 4 K$^2$

$\psi_{\theta,STEM}$ DJF variance = 1 K$^2$
Conclusions

• The ‘mean’ atmospheric circulation is highly sensitive to the averaging method.

• The STEM circulation extends the TEM circulation to an arbitrary coordinate system by taking advantage of additional eddy statistics.

• The TEM circulation corresponds to the small variance limit of the STEM circulation.
Some thoughts on CMG:

- The need for new mathematical ideas in geoscience includes the development of new conceptual and theoretical framework.

- Water vapor and clouds remain a central problem in atmospheric/climate sciences.

- Statistical and stochastic approaches could be potentially very useful in atmospheric and oceanic science (for parameterization, data assimilation and parameter estimation).

- Progress is often serendipitous: it hard to predict which mathematical tool will solve a given physical problems. Active collaborations are keys to sustain successful exchange of ideas.